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The analytic regularization zeta function method and the cut-off method in the Casimir effect

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Abstract. The zero point energy associated with a Hermitian massless scalar field in the presence of perfectly reflecting plates in a 3D flat space-time is discussed. A new technique to unify two different methods—the zeta function and a variant of the cut-off method—used to obtain the so-called Casimir energy is presented, and the proof of the analytic equivalence between both methods is given.

1. Introduction

In a previous paper [1] we introduced a new technique for comparing two usual methods of obtaining the Casimir energy, namely the cut-off method [2] and the zeta function method [3-5]. Using this approach we proved the analytic equivalence between these two methods in a 2D spacetime. The purpose of this work is to extend this previous result to a higher dimension.

The problem of the renormalization of ill defined quantities leading to a physically significant result is a fundamental and ubiquitous question of quantum field theory. Although many regularization methods have been employed, a proof that all these different methods lead to the same result is still lacking [6, 7].

The classical example of an ill defined quantity is that of the zero point energy of a quantum field in a flat spacetime. The Wick normal ordering procedure may elude this divergence. However, Casimir showed that this procedure is not adequate for the study of fields in the presence of surfaces where the fields satisfy boundary conditions. Using the idea that, although formally divergent, the zero point energy can suffer a finite change if the physical configuration is modified, he derived a finite result for the energy of the vacuum state of an electromagnetic field in the presence of conducting parallel plates. This method can be summarized in the following steps: a complete set of mode solutions and the respective eigenfrequencies of the classical wave equation satisfying appropriate boundary conditions is found; the divergent zero point energy of the quantized field is regularized by means of a cut-off function and is then renormalized using auxiliary configurations which are added and subtracted.

Subsequently other methods, like the Green function method [8-10], the dimensional regularization method [11, 12] and the zeta function method [3-5] were employed to obtain a finite result for the vacuum energy. Even in the well studied case of the Casimir energy, however, a proof of the equivalence between some of these different techniques was not available. In this article an analytic proof of the equivalence between the zeta function and the cut-off method for obtaining the Casimir energy of a scalar field confined in rectangular cavities satisfying Dirichlet boundary conditions is presented for the case $D=3$ (3D space-time). The generalization to $D>3$ and fields of higher spin is straightforward.

This paper is organized as follows: in section 2 the zeta function regularization method is briefly presented. In section 3 the exponential cut-off method is carefully studied. In section 4 the zeta function method is interpreted as an 'algebraic' cut-off method. In section 5 the unification between these two methods is achieved using the mixed cut-off procedure. The equivalence between these methods is obtained as a consequence of the analyticity of a certain complex function of two variables. Conclusions are given in section 6.

In this paper we use $\hbar = c = 1$.

2. The Casimir energy obtained using the zeta function method

For the massless scalar field confined in a 2D rectangular box satisfying Dirichlet boundary conditions the eigenfrequencies are given by

$$\omega_{nm} = \left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2\right]^{1/2} \qquad n, m = 1, 2, 3, \ldots \tag{2.1}$$

where L_1, L_2 are the lengths of the sides of the box.

The zero point energy is

$$E(L_1, L_2) = \frac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm} \tag{2.2}$$

where ω_{nm} is given by equation (2.1). This expression is divergent and can be written as

$$E_\zeta(L_1, L_2, s) = \frac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm}^{-2S} \tag{2.3}$$

for $s=-1/2$.

The expression (2.3) is analytic for $\mathrm{Re}(s)>1$. The zeta function method consists in evaluating the analytic continuation of this function at the point $s=-1/2$, thereby obtaining a finite result. Algebraic manipulations of equation (2.3), using equation (2.1), give

$$E_\zeta(L_1, L_2, s) = \frac{1}{8} A\left(\left(\frac{\pi}{L_1}\right)^2, \left(\frac{\pi}{L_2}\right)^2; 2s\right) - \frac{1}{4}\left(\left(\frac{L_1}{\pi}\right)^{2s} + \left(\frac{L_2}{\pi}\right)^{2s}\right)\zeta(2s) \tag{2.4}$$

where $\zeta(2s)$ is the Riemann zeta function and $A(a, b; 2s)$ is the Epstein zeta function defined as [4, 13]

$$A(a_1, a_2, \ldots, a_k; 2s) = \sum_{n_1, n_2, \ldots, n_k=-\infty}^{\infty}{}' (a_1 n_1^2 + a_2 n_2^2 + \ldots + a_k n_k^2)^{-s}.$$

The prime sign in the summation means that the term $n_1 = n_2 = \ldots = n_k = 0$ is to be excluded. So $E_\zeta(L_1, L_2; s)$ is analytic in $s \in \mathbb{C}\backslash\{1/2, 1\}$ and the evaluation of $E_\zeta(L_1, L_2; -1/2)$ gives the Casimir energy $U(L_1, L_2)$,

$$U(L_1, L_2) = \frac{\pi}{48}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) - \frac{L_1 L_2}{32\pi} \sum_{p,q=-\infty}^{\infty}{}' (p^2 L_1^2 + q^2 L_2^2)^{-3/2}. \tag{2.5}$$

To obtain the Casimir energy given by equation (2.5) through this method there is apparently no need for a renormalization scheme and a finite result comes out automatically.

The basic idea employed by Casimir is that although the zero point energy is divergent, changes in the configuration lead to a finite shift in the total energy. In the exponential cut-off method, in order to evaluate this shift the total energy is regularized before the subtraction from the energy of a reference configuration. This total energy is obtained by adding the regularized zero point energy of the field inside and outside the cavity. This approach seems very natural if we are dealing with a system in which there is field inside and outside the cavity (let us call this system a 'box' (figure 1*a*)). If we suppose that outside the cavity there is no field (as in the bag model or for the electromagnetic field in a bubble surrounded by a perfectly conducting material), there is only the contribution of the field inside the cavity (let us call this system a 'bubble' (figure 1*b*)). As was briefly discussed in this section, the zeta function method apparently does not take into account the field outside the cavity. The two configurations, bubble and box, seem different and one question arises. Which one of these configurations did the zeta method discussed in this section treat? This point will be investigated later.

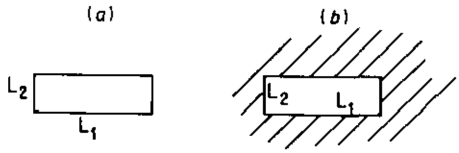

Figure 1. (*a*) The box configuration. (*b*) The bubble configuration.

3. The exponential cut-off method

The divergent expression given by equation (2.2) can be regularized using an exponential cut-off function such as

$$e^{-\lambda\omega_{nm}}. \tag{3.1}$$

The regularized energy is then

$$E_e(L_1, L_2, \lambda) = \tfrac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm}\, e^{-\lambda\omega_{nm}} \qquad \mathrm{Re}(\lambda) > 0. \tag{3.2}$$

The function given by equation (3.2) is analytic for $\mathrm{Re}(\lambda) > 0$ but divergent at $\lambda = 0$. A renormalization procedure is thus required to enable one to take the limit $\lambda \to 0^+$ without divergences. This method was employed by Casimir, Fierz [14], Boyer [15] and others using auxiliary configurations (taking into account the field outside the cavity) in order to obtain a finite result for the case of parallel plates.

Let us define a function $h(a, b, u)$ which will be of use throughout this paper:

$$h(a, b, u) = \sum_{n,m=1}^{\infty} \exp\left[-u\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2}\right] \qquad a, b > 0. \tag{3.3}$$

Then the regularized energy given by equation (3.2) can be expressed as

$$E_e(L_1, L_2, \lambda) = -\frac{1}{2}\frac{\partial}{\partial\lambda} h(L_1, L_2, \lambda). \tag{3.4}$$

Adding and subtracting terms in equation (3.3) in order to get a double summation in $n, m \in \mathbb{Z}$, performing this summation using the Poisson summation formula, integrating this result in polar coordinates and using the fact that the two simple summations which are introduced are geometric series, we get the following expression for the function h:

$$h(a, b, u) = \frac{ab}{2\pi} u^{-2} + \frac{ab}{2\pi} u \sum_{p,q=-\infty}^{\infty}{}' (u^2 + 4p^2a^2 + 4q^2b^2)^{-3/2}$$
$$-\frac{1}{2}\frac{1}{e^{\pi u/a}-1} - \frac{1}{2}\frac{1}{e^{\pi u/b}-1} - \frac{1}{4}. \tag{3.5}$$

Defining $r_0 = \min\{a, b\}$ we see from equation (3.5) that $h(a, b, u)$ is analytic for $0 < |u| < 2r_0$, the point $u = 0$ is a second-order pole and the negative powers portion of the Laurent series expansions of h around $u = 0$ is given by

$$h_{\text{polar}}(a, b, u) = \frac{ab}{2\pi} u^{-2} - \frac{a+b}{2\pi} u^{-1}.$$

Substituting equation (3.5) in equation (3.4) the regularized energy becomes

$$E_e(L_1, L_2, \lambda)$$
$$= \frac{L_1 L_2}{2\pi}\lambda^{-3} - \frac{1}{4\pi} B_0 (L_1 + L_2)\lambda^{-2} + \frac{\pi}{8} B_2\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$
$$+ \frac{L_1 L_2}{4\pi} \sum_{p,q=-\infty}^{\infty}{}' (\lambda^2 + 4p^2a^2 + 4q^2b^2)^{-3/2} + \lambda^2 g_1(\lambda) \tag{3.6}$$

where B_0 and B_2 are Bernoulli numbers and $g_1(\lambda)$ is analytic in $|\lambda| < 2\min\{L_1, L_2\}$.

The two divergent terms in equation (3.6) are proportional to the 'volume' and to the 'perimeter' of the cavity; thus following the Casimir approach we need to add and subtract auxiliary configurations in order that:

(i) The final result is a difference between 'isovolumetric' and 'isoperimetric' configuration sets.

(ii) The auxiliary configurations should not give spurious contributions to the finite renormalized energy.

(iii) The energy of the modes of the field inside and outside the cavity are present in the total energy.

In order to satisfy prescription (iii) it is necessary to employ nested boxes and to make the lengths of the exterior box tend to infinity. This approach was employed by Boyer [15, 16] using nested spherical shells. In this paper prescription (iii) will not be used and only rectangular cavities will be employed. Therefore we will work with a variant of the exponential cut-off method, following only prescriptions (i) and (ii).

Prescription (ii) is achieved if the distance between the opposite sides of the auxiliary cavities becomes infinite, so that the field inside these auxiliary boxes tends to the free, unconstrained field.

A naïve procedure to obtain the Casimir energy in the rectangular cavity case would be to define

$$U(L_1, L_2) = \lim_{\substack{\lambda\to 0\\ R,R'\to\infty}} E_e(L_1, L_2, \lambda) + E_e(R - L_1, L_2, \lambda) + E_e(L_1, R' - L_2, \lambda) + E_e(R - L_1, R' - L_2, \lambda) - 4E_e(R/2, R'/2, \lambda). \tag{3.7}$$

This can be visualized by figures 2(a) and (b).

The problem of this renormalization is that we are adding 'plates' to the initial configuration. The field inside the cavities of sides $(R - L_1, L_2)$ and $(L_1, R' - L_2)$ will never tend to the free unconstrained field as R, $R'\to\infty$ as in the two-plate Casimir approach, and prescription (ii) is thus not satisfied. This (equation 3.7) renormalized energy must be corrected by removing this 'plate' effect. This can be done by means of the following renormalization:

$$\begin{aligned} Y(L_1, L_2, R, R', \lambda) &= E_e(L_1, L_2, \lambda) + E_e(R - L_1, L_2, \lambda) + E_e(L_1, R' - L_2, \lambda) \\ &\quad + E_e(R - L_1, R' - L_2, \lambda) - 4E_e(R/2, R'/2, \lambda) \\ &\quad - [E_e(L_1, R' - L_2, \lambda) + E_e(R - L_1, R' - L_2, \lambda) - 2E_e(R/2, R' - L_2, \lambda)] \\ &\quad - [E_e(R - L_1, L_2, \lambda) + E_e(R - L_1, R' - L_2, \lambda) \\ &\quad - 2E_e(R - L_1, R'/2, \lambda)]. \end{aligned} \tag{3.8}$$

Then

$$U(L_1, L_2) = \lim_{\substack{\lambda\to 0\\ R,R'\to\infty}} Y(L_1, L_2, R, R', \lambda). \tag{3.9}$$

The evaluation of $U(L_1, L_2)$ by equation (3.9) gives the same result as that obtained using equation (2.5). It is easy to see that if we use the following auxiliary configurations (illustrated in figures 3a and b), which are 'isovolumetric' and 'isoperimetric', the finite

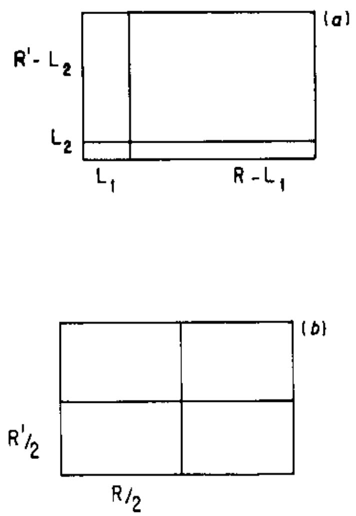

Figure 2. Set of configurations employed to obtain $U(L_1, L_2)$ of equation (3.7).

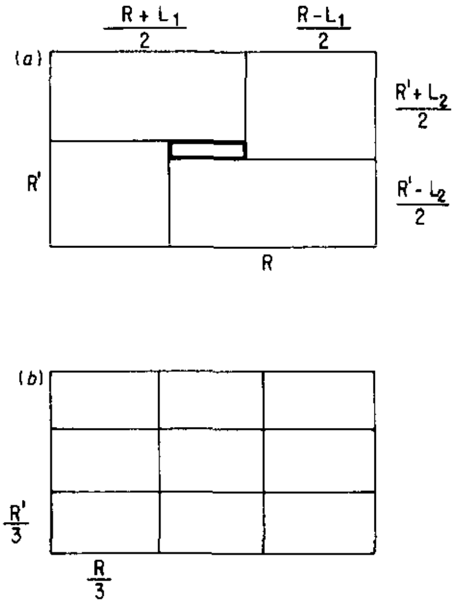

Figure 3. Set of configurations that gives the same $U(L_1, L_2)$ as that obtained by equations (2.5) and (3.9).

contribution of the auxiliary cavities to the Casimir energy vanish as R, $R' \to \infty$. This strange configuration gives the same result as that obtained from equation (3.9), as expected.

4. The zeta function method as an 'algebraic' cut-off method

A regularized energy can be obtained from equation (2.2) using an 'algebraic' cut-off:

$$\omega_{nm}^{-\sigma} \tag{4.1}$$

and the regularized energy becomes

$$E_{\mathrm{a}}(L_1, L_2, \sigma) = \tfrac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm}\omega_{nm}^{-\sigma} \qquad \mathrm{Re}(\sigma) > 3 \tag{4.2}$$

which is convergent and analytic for $\mathrm{Re}(\sigma) > 3$. Of course $E_{\mathrm{a}}(L_1, L_2, \sigma) = E_\zeta(L_1, L_2, s = (\sigma - 1)/2)$, and using this relation and equation (2.4) we get

$$E_{\mathrm{a}}(L_1, L_2, \sigma) = \frac{1}{8} A\left(\left(\frac{\pi}{L_1}\right)^2, \left(\frac{\pi}{L_2}\right)^2; \sigma - 1\right) - \frac{1}{4}\left(\left(\frac{L_1}{\pi}\right)^{\sigma-1} + \left(\frac{L_2}{\pi}\right)^{\sigma-1}\right)\zeta(\sigma - 1). \tag{4.3}$$

If $\sigma > 3$, this cut-off works well and we get a finite energy. As in any cut-off method, we wish to take the limit $\sigma \to 0$ starting from $\sigma > 3$. Equation (4.3) defines an analytic function in $\sigma \in \mathbb{C}\backslash\{2, 3\}$ (see figure 4).

It is interesting to note that equation (4.3), when evaluated at $\sigma = 0$, gives the Casimir energy derived in section 2. This later result (of section 2) is based on the use

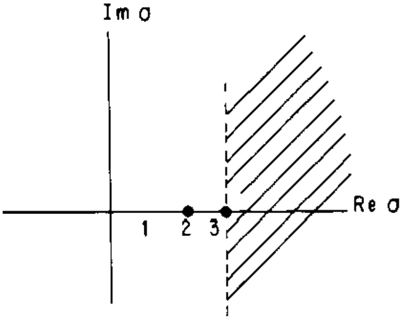

Figure 4. The summation of equation (4.2) is convergent in the shadowed region, while the analytic continuation of this function is a meromorphic function with its poles indicated.

of the analytic continuation of a complex function. Although it seems to be quite obvious, we wish to point out that analytic continuations:

(i) Are to be performed in open connected domains.

(ii) Make use of paths (entirely contained in the domain) in order to extend the function which is initially defined only in a subset of the domain.

Since we are dealing with the zeta method as a cut-off method, we will move σ (our regularization parameter) only along the real axis, and certain procedures related to the physics of the problem will be employed, namely, sum and subtraction of auxiliary cavities aimed at eliminating divergences and satisfying prescriptions (i) and (ii) of section 3.

A careful study of equation (4.3) leads us to the expression

$$E_{\mathrm{a}}(L_1, L_2, \sigma) = G_1(L_1, L_2, \sigma) + \frac{1}{4\Gamma((\sigma-1)/2)}\left(\frac{L_1 L_2}{\pi(\sigma-3)} - \frac{L_1+L_2}{\pi^{3/2}(\sigma-2)}\right) \tag{4.4}$$

where $G_1(L_1, L_2, \sigma)$ is analytic in the whole σ-complex plane. As we move along the real axis from $\sigma > 3$ towards $\sigma = 0$, we find, first, a divergence proportional to the 'volume' of the cavity and, after that, a divergence proportional to the 'perimeter' of the cavity. Again, it is clear that it is necessary to use auxiliary configurations with 'isovolumetric' and 'iosperimetric' subtractions in order to eliminate the divergences along the path. If we take the auxiliary configurations as in equation (3.8) or as in figures 3(a) and (b) (therefore satisfying prescriptions (i) and (ii) of section 3) the result will be the same as that of section 2, since the auxiliary configurations will not disturb the value of the analytic continuation of equation (4.3) at the point $\sigma = 0$ in the limit R, $R' \to \infty$. Therefore the zeta function method employed in section 3 (for rectangular cavities) is analytically equivalent to the 'algebraic' cut-off method presented in this section.

Since the exponential cut-off is a strong factor of convergence, the regularized energy $E_{\mathrm{e}}(L_1, L_2, \lambda)$ becomes singular only when $\lambda \to 0^+$. Being a weaker factor of convergence, the algebraic cut-off scatters the singularities of the regularized energy $E_{\mathrm{a}}(L_1, L_2, \sigma)$ along the path towards the origin, leaving the origin itself free from singularities when we take the analytic continuation.

5. The mixed cut-off as a tool for unification

The procedure of sections 3 and 4 can be unified by the use of a 'mixed' cut-off function

$$\omega_{nm}^{-\sigma}\, \mathrm{e}^{-\lambda\omega_{nm}}. \tag{5.1}$$

In this case the regularized energy is

$$E_{\mathrm{M}}(L_1, L_2, \lambda, \sigma) = \tfrac{1}{2} \sum_{n,m=1}^{\infty} \omega_{nm}\omega_{nm}^{-\sigma} \, \mathrm{e}^{-\lambda\omega_{nm}}$$
$$\mathrm{Re}(\lambda) > 0 \qquad \sigma \in \mathbb{C} \qquad \text{or} \qquad \mathrm{Re}(\lambda) = 0 \qquad \mathrm{Re}(\sigma) > 3. \tag{5.2}$$

The regularized energy given by equation (5.2) as a function of λ and σ is analytic in $\mathrm{Re}(\lambda) > 0$, $\sigma \in \mathbb{C}$ and is continuous in $\mathrm{Re}(\lambda) \geq 0$, $\mathrm{Re}(\sigma) > 3$. In $\lambda = 0$, as a function of σ, it is possible to continue it analytically in $\sigma \in \mathbb{C}\backslash\{2, 3\}$.

Of course in view of equations (3.2), (4.2) and (5.2)

$$E_{\mathrm{e}}(L_1, L_2, \lambda) = E_{\mathrm{m}}(L_1, L_2, \lambda, \sigma = 0)$$
$$E_{\mathrm{a}}(L_1, L_2, \sigma) = E_{\mathrm{m}}(L_1, L_2, \lambda = 0, \sigma)$$

and $\lim_{\sigma,\lambda\to 0} E_{\mathrm{m}}(L_1, L_2, \lambda, \sigma)$ is not defined.

The regularized energy thus obtained in equation (5.2) can be renormalized using the same procedures as in sections 3 and 4: addition and subtraction of auxiliary configurations. This procedure can be formalized as follows. Define

$$Y(L_1, L_2, R, R', \lambda, \sigma) = \sum_{i=1}^{N} E_{\mathrm{m}}(L_{1i}, L_{2i}, \lambda, \sigma) - \sum_{i=N+1}^{2N} E_{\mathrm{m}}(L_{1i}, L_{2i}, \lambda, \sigma) \tag{5.3}$$

where the original cavity with lengths (L_1, L_2) appears as (L_{11}, L_{21}). The other L_{ki}, $k = 1, 2$, $i = 2, 3 \ldots 2N$ are monotonous functions of R, R' in such a way that

$$\lim_{R, R' \to \infty} L_{ki} = \infty \qquad L_{ki} \geq L_{k1} \qquad i \neq 1. \tag{5.4}$$

Since we want 'isovolumetric' and 'isoperimetric' subtraction, it must be imposed that

$$\sum_{i=1}^{N} L_{1i} + L_{2i} = \sum_{i=N+1}^{2N} L_{1i} + L_{2i} \tag{5.5}$$

and

$$\sum_{i=1}^{N} L_{1i}L_{2i} = \sum_{i=N+1}^{2N} L_{1i}L_{2i}. \tag{5.6}$$

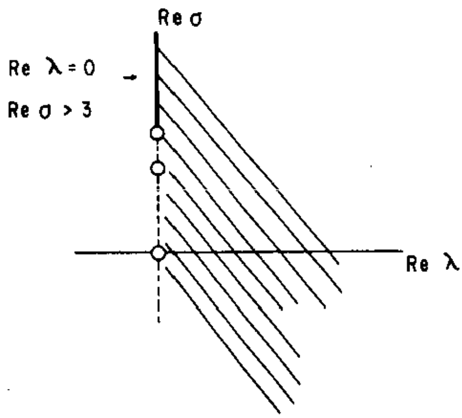

Figure 5. The summation of equation (5.2) converges to an analytic function (in λ, σ) in the shadowed region. It also converges in $\mathrm{Re}(\lambda) = 0$, $\mathrm{Re}(\sigma) > 3$. The points $\sigma = 2$, 3 are poles of the analytic continuation of $E_{\mathrm{m}}(L_1, L_2, \lambda = 0, \sigma)$.

In section 4 it was proved that the zeta function method is equivalent to the algebraic cut-off method, which is a particular case of the use of the mixed cut-off method. This happens when we evaluate

$$U_{\text{alg}}(L_1, L_2) = \lim_{\substack{\sigma \to 0^+ \\ R, R' \to \infty}} Y(L_1, L_2, R, R', \lambda = 0, \sigma) \tag{5.7}$$

and the Casimir energy derived by the exponential cut-off method is given by

$$U_{\text{exp}}(L_1, L_2) = \lim_{\substack{\lambda \to {}^+0 \\ R, R' \to \infty}} Y(L_1, L_2, R, R', \lambda, \sigma = 0). \tag{5.8}$$

We claim that the 'isovolumetric' and 'isoperimetric' subtraction performed in equation (5.3) renders the function $Y(L_1, L_2, R, R', \lambda, \sigma)$ analytic in $|\lambda| < \rho_0$, $\sigma \in \mathbb{C}$ for some $\rho_0 > 0$. Consequently, equations (5.7) and (5.8) give the same result and the two methods—cut-off and zeta function—are analytically equivalent. Now we conclude the proof demonstrating the analyticity of $Y(L_1, L_2, R, R', \lambda, \sigma)$ in a domain $|\lambda| < \rho_0$, $\sigma \in \mathbb{C}$ for some $\rho_0 > 0$.

Let us call

$$\Omega(\rho_0) = \{\lambda \in \mathbb{C}; |\lambda| < \rho_0\} \times \{\sigma \in \mathbb{C}\} \qquad \text{for } \rho_0 > 0. \tag{5.9}$$

Using an integral representation of the Γ function, the regularized energy given by equation (5.2) can be expressed for $\text{Re}(\lambda) > 0$ or $\text{Re}(\lambda) = 0$ and $\text{Re}(\sigma) > 3$ as

$$E_{\text{m}}(L_1, L_2, \lambda, \sigma) = \frac{1}{2} \frac{1}{\Gamma(\sigma - 1)} \int_0^\infty \mathrm{d}x\, x^{\sigma-2} \sum_{n, m=1}^{\infty} \mathrm{e}^{-(\lambda + x)\omega_{nm}}$$
$$\text{Re}(\lambda) > 0 \qquad \text{or} \qquad \text{Re}(\lambda) = 0 \qquad \text{and} \qquad \text{Re}(\sigma) > 3. \tag{5.10}$$

Using equations (2.1) and (3.3) and splitting the above integral in ρ_0 we get

$$E_{\text{m}}(L_1, L_2, \lambda, \sigma) = \frac{1}{2} \frac{1}{\Gamma(\sigma - 1)} \int_0^{\rho_0} \mathrm{d}x\, x^{\sigma-2} h(L_1, L_2, \lambda + x) + g_2(L_1, L_2, \lambda, \sigma)$$
$$\text{Re}(\lambda) > 0 \qquad \text{or} \qquad \text{Re}(\lambda) = 0 \qquad \text{and} \qquad \text{Re}(\sigma) > 3 \tag{5.11}$$

where $g_2(L_1, L_2, \lambda, \sigma)$ is analytic in $\Omega(\rho_0)$.

Defining

$$H(L_1, L_2, R, R', u) = \sum_{i=1}^{N} h(L_{1i}, L_{2i}, u) - \sum_{i=N+1}^{2N} h(L_{1i}, L_{2i}, u) \tag{5.12}$$

and

$$G_2(L_1, L_2, R, R', \lambda, \sigma) = \sum_{i=1}^{N} g_2(L_{1i}, L_{2i}, \lambda, \sigma) - \sum_{i=N+1}^{2N} g_2(L_{1i}, L_{2i}, \lambda, \sigma) \tag{5.13}$$

then from equations (5.3) and (5.11)-(5.13) we have

$$\begin{aligned} Y(L_1, L_2, R, R', \lambda, \sigma) &= \frac{1}{2} \frac{1}{\Gamma(\sigma - 1)} \int_0^{\rho_0} \mathrm{d}x\, x^{\sigma-2} H(L_1, L_2, \lambda + x) \\ &\quad + G_2(L_1, L_2, R, R', \lambda, \sigma). \end{aligned} \tag{5.14}$$

Since $G_2(L_1, L_2, R, R', \lambda, \sigma)$ is a sum of analytic functions in $(\lambda, \sigma) \in \Omega(\rho_0)$, this function is analytic in the same domain.

$H(L_1, L_2, R, R', u)$ is a sum of $2N$ functions, each one analytic in $0<|u|<2\,\mathrm{Min}\{L_{1i}, L_{2i}\}$, with a second-order pole at $u=0$. Then if we take

$$\rho_0 = \mathrm{Min}\{L_1, L_2\} \tag{5.15}$$

thus from equation (5.4) it follows that $H(L_1, L_2, R, R', u)$ is analytic at $0<|u|<2\rho_0$ and has, at worst, a second-order pole at $u=0$. Using equation (5.12), the polar portion of each $h(L_{1i}, L_{2i}, u)$ derived in section 3 and the restrictions imposed upon L_{ki} by equations (5.5) and (5.6), we find that the coefficients of the negative portion of the Laurent series of $H(L_1, L_2, R, R', u)$ (around $u=0$) vanish. Then $H(L_1, L_2, R, R', u)$ is analytic at $|u|<2\rho_0$. Thus using equation (5.9) and the properties of the Γ function we see that $Y(L_1, L_2, R, R', \lambda, \sigma)$ has an analytic continuation in $(\lambda, \sigma) \in \Omega(\rho_0)$, as we claimed.

6. Conclusions

In this paper we developed a consistent method to unify two hitherto unrelated regularization methods employed to obtain the Casimir energy, the zeta function method and the exponential cut-off method.

Rectangular cavities with Dirichlet boundary conditions in a 3D space-time ($D=3$) were studied. We proved the analytic equivalence between the zeta function method and a variant of the exponential cut-off method for these configurations. The generalization for higher dimensional space-times is straightforward.

It is important to note that it was showed that the zeta function method is analytically equivalent to the 'algebraic' cut-off method of section 4. This second method performs configurations subtraction satisfying prescriptions (i) and (ii) (section 3). Therefore we can say that the zeta function method performs 'virtual' configurations subtraction (satisfying prescriptions (i) and (ii)). This equivalence, together with the analytic equivalence between the algebraic and the exponential cut-off method (proved in section 4), gives a proof of the analytic equivalence between the zeta function method and the variant of the cut-off method of section 3. Once again we speak of a variant of the exponential cut-off method because prescription (iii) was not followed and none of the auxiliary configurations employed reproduced in any sense the geometry of the space outside the original cavity. In Casimir's [2] original work and in Fierz [14] and Boyer's [15] papers, once the region outside the plates is the union of two simple connected domains (in fact, two semi-spaces) this kind of problem does not exist and therefore the contribution of the exterior modes are cancelled out in the renormalization procedure. In the case of spherical shells [15–17] the cut-off method has been employed with the use of concentric auxiliary cavities; then one of the auxiliary cavities reproduces the geometry of the space outside the original shell. For $D-2$ dimensional parallel plates in a D-dimensional space-time such problems do not appear and so it is straightforward to prove the analytic equivalence between the zeta and the exponential cut-off method for these configurations by means of the mixed cut-off method.

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